HyperKähler Contact Distribution in 3-Sasakain Manifolds

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Abstract. In this paper, the HyperKähler contact distribution of a 3-Sasakian manifold is studied. To analyze the curvature properties of this distribution, the special metric connection $\bar{\nabla}$ is defined. This metric connection is completely determined by HyperKähler contact distribution. We prove that HyperKähler contact distribution is of constant holomorphic sectional curvatures if and only if its 3-Sasakian manifold is of constant φ_α-sectional curvatures. Moreover, it is shown that there is an interesting relation between the sectional curvatures of φ_α-planes on TM of metric connection $\bar{\nabla}$ and the Levi-Civita connection.

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1. Introduction

Historically Sasakian structures grew out of research in contact manifolds. In 1960, Sasaki [14] introduced a geometric structure related to an almost contact structure. This geometry became known as Sasakian geometry and has been studied extensively ever since. In the 1960's, the Sasakian structure was studied extensively as an odd dimensional analogue of Kähler spaces [18].

Afterwards, by developing the Kähler structure to Quaternionic spaces the concept of Quaternionic Kähler and HyperKähler manifolds were introduced. HyperKähler manifolds are special class of Kähler manifolds. They can be thought of as quaternionic analogues of Kähler manifolds while a quaternionic Kähler manifold needs not to be a Kähler one in general. In recent years quaternionic Kähler and HyperKähler manifolds have received a great deal of attention. They appear and play an important role in many different areas of mathematics and mathematical physics [1, 4, 5, 7, 9]. The 3-Sasakian manifolds, which have a close relation to both HyperKähler and Quaternionic Kähler manifolds, first appeared in a paper by Kuo in [11]. In

1970 three more papers [12, 15, 17] were published in the Japanese literature discussing Sasakian 3-structures. Later on, in 1973 Ishihara [8] had shown that if the distribution formed by the three Killing vector fields which define the 3-Sasakian structure is regular then the space of leaves is a quaternionic Kähler manifold.

We assume that $(M, \varphi_\alpha, \eta^\alpha, \xi_\alpha, g)$ is a 3-Sasakian manifold and $\mathbf{H}$ denotes the transverse distribution to the Riemannian foliation $< \xi_1, \xi_2, \xi_3 >$ with respect to the metric g in the tangent bundle TM. The purpose of this paper is to study the geometric properties of the distribution $\mathbf{H}$ as well as the tangent bundle of a manifold such as curvature tensor, sectional curvature and Ricci tensor. Moreover, we investigate the close relation of φ_α-sectional curvatures of M and holomorphic sectional curvatures of $\mathbf{H}$. We can refer to [2] as an analogue work on Sasakian manifolds.

Aiming at our purpose, we organized this paper as follows. In section 2, we present some basic notations and definitions which are needed in the following sections. In section 3, the new linear connection $\bar{\nabla}$ is introduced in the terms of Levi-Civita connection. We show that it is a metric connection and completely determined by $\mathbf{H}$. Moreover, the φ_α structures are parallel with respect to the new connection $\bar{\nabla}$ on $\mathbf{H}$. In section 4, we present the HyperKähler properties of the distribution $\mathbf{H}$ and call it the HyperKähler contact distribution. In this section, the curvature and Ricci tensor of HyperKähler contact distribution $\mathbf{H}$ is defined with respect to the metric connection $\bar{\nabla}$. Finally, in this section, we prove some theorems that show the curvature properties of this distribution and its close geometric relation with the 3-Sasakian manifold.

2. Preliminaries and Notations

Let M be a $(2n+1)$-dimensional smooth manifold. Then, the triple structure (φ, η, ξ) on M, consisting of the $(1,1)$-form φ, non-vanishing vector field ξ and the 1-form η, is called an *almost contact structure* if

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

This structure will be called a *contact structure* if

$$\eta \wedge (d\eta)^n \neq 0.$$

The manifold M^{2n+1} with the almost contact or contact structure (φ, η, ξ) is called almost contact or contact manifold, respectively, and they are denoted by (M, φ, η, ξ). There was proved that every almost contact manifolds admit a compatible Riemannian metric in the following sense

$$\eta(X) = g(\xi, X), \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.1}$$

for all $X, Y \in \Gamma(TM)$. In case of contact metric manifolds, the fundamental 2-form Ω defined by

$$\Omega(X, Y) = g(X, \varphi Y), \qquad \forall X, Y \in \Gamma(TM)$$

coincides with $d\eta$.

Let ∇ be the Levi-Civita connection with respect to the metric g on the contact manifold (M,φ,η,ξ,g). Then, the (almost) contact metric manifold (M,φ,η,ξ,g) is a *Sasakian manifold* if

$$(\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X.$$

If (M,φ,η,ξ,g) is a Sasakian manifold then the following equations are satisfied

$$\begin{gathered} \nabla_X\xi = -\varphi X, \quad R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \\ R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \quad S(X,\xi) = 2n\eta(X), \end{gathered} \tag{2.2}$$

where R and S are curvature tensor and Ricci curvature, respectively, given by following formulas

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{2.3}$$

$$S(X,Y) = \sum_{i=1}^{2n+1} R(E_i,X,E_i,Y) := \sum_{i=1}^{2n+1} g(R(E_i,X)Y,E_i), \tag{2.4}$$

where E_i are orthonormal local vector fields on (M,g) [13, 16].

Let (M,g) be a smooth Riemannian manifold of dimension $4n+3$. The manifold (M,g) is called a 3-*Sasakian manifold* when it is endowed with three Sasakian structures $(M,\varphi_\alpha,\eta^\alpha,\xi_\alpha,g)$ for $\alpha=1,2,3$, satisfying the following relations

$$\begin{gathered} \varphi_\theta = \varphi_\beta\varphi_\gamma - \eta^\gamma\otimes\xi_\beta = -\varphi_\gamma\varphi_\beta + \eta^\beta\otimes\xi_\gamma, \\ \xi_\theta = \varphi_\beta\xi_\gamma = -\varphi_\gamma\xi_\beta, \quad \eta^\theta = \eta^\beta\circ\varphi_\gamma = -\eta^\gamma\circ\varphi_\beta, \end{gathered}$$

for all even permutation (β,γ,θ) of $(1,2,3)$. For more details about the above definitions see [3].

Let ξ be the distribution spanned by the three global vector fields $<\xi_1,\xi_2,\xi_3>$. By means of $\nabla_X\xi_\alpha = -\varphi_\alpha X$ for $\alpha=1,2,3$, one can prove the integrability of ξ as follows

$$[\xi_\alpha,\xi_\beta] = \nabla_{\xi_\alpha}\xi_\beta - \nabla_{\xi_\beta}\xi_\alpha = 2\xi_\gamma, \tag{2.5}$$

for all even permutation (α,β,γ) of $(1,2,3)$. Therefore, ξ defines a 3-dimensional foliation on M. Moreover, the equations $\nabla_{\xi_\alpha}g = 0$ for $\alpha=1,2,3$ show that the foliation ξ is a Riemannian foliation. The transverse distribution of ξ with respect to the metric g is denoted by $\mathbf{H}$ and it is given by $\mathbf{H} = \cap_{\alpha=1}^{3} ker(\eta^\alpha)$. The distribution $\mathbf{H}$ is a $4n$-dimensional distribution on M and it decomposes the tangent bundle TM as follows

$$TM = \mathbf{H}\oplus\xi.$$

In a 3-Sasakian manifold, the distribution $\mathbf{H}$ is never integrable and in the sequel we call it 3-*contact distribution*.

3. H-Connection of 3-Contact Distribution

Consider the foliation ξ and the distribution $\mathbf{H}$ on a 3-Sasakian manifold M. We can choose a local coordinate system by means of foliated charts such that

$$\forall \mathbf{x} \in M \quad \mathbf{x} = (z^1, z^2, z^3, x^1, ..., x^{4n}),$$

with $\xi =< \frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2}, \frac{\partial}{\partial z^3} >$. This local frame is adapted to the foliation ξ and one can make the local basis $\{\frac{\delta}{\delta x^1}, ..., \frac{\delta}{\delta x^{4n}}\}$ of $\mathbf{H}$ orthogonal to ξ with respect to the metric g from the basis $\{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^{4n}}\}$ as follows

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \eta_i^\alpha \xi_\alpha, \quad i = 1, ..., 4n \quad \alpha = 1, 2, 3$$

where $\eta_i^\alpha = \eta^\alpha(\frac{\partial}{\partial x^i})$. Consider local basis

$$\{\xi_1, \xi_2, \xi_3, \frac{\delta}{\delta x^1}, ..., \frac{\delta}{\delta x^{4n}}\}, \tag{3.1}$$

of TM, then the Riemannian metric g in this local basis will have the presentation as follows

$$g := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g_{ij} \end{pmatrix} \tag{3.2}$$

where $g_{ij} = g(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})$.

Now, we evaluate the Lie brackets of the local vector fields in (3.1). First, by a direct calculation we obtain

$$[\frac{\delta}{\delta x^i}, \xi_\alpha] = \xi_\alpha(\eta_i^\beta)\xi_\beta - \eta_i^\beta[\xi_\beta, \xi_\alpha], \quad [\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}] = \left(\frac{\delta \eta_i^\alpha}{\delta x^j} - \frac{\delta \eta_j^\alpha}{\delta x^i}\right)\xi_\alpha. \tag{3.3}$$

Then, the Lie brackets of ξ_1,ξ_2 and ξ_3 are given by (2.5). By means of

$$-\eta^\beta([\frac{\delta}{\delta x^i}, \xi_\alpha]) = 2d\eta^\beta(\frac{\delta}{\delta x^i}, \xi_\alpha) = 2g(\frac{\delta}{\delta x^i}, \varphi_\beta(\xi_\alpha)) = 0,$$

and

$$\Omega_{ij}^\alpha = \Omega^\alpha(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = d\eta^\alpha(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = -\frac{1}{2}\eta^\alpha([\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}]),$$

the Eq. (3.3) will be appeared as follows

$$[\frac{\delta}{\delta x^i}, \xi_\alpha] = 0, \quad [\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}] = \left(\frac{\partial \eta_i^\alpha}{\partial x^i} - \frac{\partial \eta_j^\alpha}{\partial x^i}\right)\xi_\alpha = -2\Omega_{ij}^\alpha \xi_\alpha. \tag{3.4}$$

Now, we consider tensor fields $h_{\alpha\beta}$ given by

$$h_{\alpha\beta}(X) = \frac{1}{2}\left(\mathcal{L}_{\xi_\alpha}\varphi_\beta\right)(X),$$

for $\alpha, \beta = 1, 2, 3$. These tensor fields on the 3-Sasakian manifold $(M, \varphi_\alpha, \eta^\alpha, \xi_\alpha, g)$ have the following more explicit expressions

$$\begin{aligned}
&h_{11}(X) = h_{22}(X) = h_{33}(X) = 0,\\
&h_{12}(X) = -h_{21}(X) = \varphi_3(X),\\
&h_{31}(X) = -h_{13}(X) = \varphi_2(X),\\
&h_{23}(X) = -h_{32}(X) = \varphi_1(X),
\end{aligned}$$

for all $X \in \Gamma TM$.

Theorem 3.1. *The Levi-Civita connection ∇ of the Riemannian metric g on the 3-Sasakian manifold $(M, \varphi_\alpha, \eta^\alpha, \xi_\alpha, g)$ has the following components with respect to basis (3.1)*

$$\begin{aligned}
&\nabla_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = F_{ij}^k \frac{\delta}{\delta x^k} - \Omega_{ij}^\alpha \xi_\alpha,\\
&\nabla_{\xi_\alpha} \frac{\delta}{\delta x^i} = \nabla_{\frac{\delta}{\delta x^i}} \xi_\alpha = \Omega_{ij}^\alpha g^{jk} \frac{\delta}{\delta x^k},\\
&\nabla_{\xi_1}\xi_2 = -\nabla_{\xi_2}\xi_1 = \xi_3,\\
&\nabla_{\xi_3}\xi_1 = -\nabla_{\xi_1}\xi_3 = \xi_2,\\
&\nabla_{\xi_2}\xi_3 = -\nabla_{\xi_3}\xi_2 = \xi_1,\\
&\nabla_{\xi_1}\xi_1 = -\nabla_{\xi_2}\xi_2 = \nabla_{\xi_3}\xi_3 = 0,
\end{aligned}$$

where

$$F_{ij}^k = \frac{g^{kh}}{2} \left\{ \frac{\delta g_{ih}}{\delta x^j} + \frac{\delta g_{jh}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^h} \right\}. \tag{3.5}$$

Corollary 3.2. [4, 5] *The foliation ξ is totally geodesic and the Riemannian metric g is bundle-like for it.*

Consider the Levi-Civita connection ∇ the on 3-Sasakian manifold (M, g). Then, we define the linear connection $\bar{\nabla}$ by

$$\bar{\nabla}_X Y = \nabla_X Y - \eta^\alpha(X) \nabla_Y \xi_\alpha - \eta^\alpha(Y) \nabla_X \xi_\alpha + \Omega^\alpha(X, Y)\xi_\alpha. \tag{3.6}$$

By direct calculations using (3.1) and (3.6) we obtain the following theorem.

Theorem 3.3. *locally, the linear connection $\bar{\nabla}$ is completely determined by the formulas*

$$\begin{aligned}
&\bar{\nabla}_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = F_{ij}^k \frac{\delta}{\delta x^k},\\
&\bar{\nabla}_{\xi_\alpha} \frac{\delta}{\delta x^i} = \bar{\nabla}_{\frac{\delta}{\delta x^i}} \xi_\alpha = 0,\\
&\bar{\nabla}_{\xi_\alpha} \xi_\beta = 0.
\end{aligned}$$

From Theorem 3.3 and (3.5), we conclude that $\bar{\nabla}$ is completely determined by Riemannian metric induced by the g on the 3-contact distribution $\mathbf{H}$. Moreover, it is easy to see that the 3-contact distribution $\mathbf{H}$ and foliation ξ are parallel with respect to the linear connection $\bar{\nabla}$. For these reasons we call $\bar{\nabla}$ the **H**-*connection* on the 3-Sasakian manifold M.

This is surprising and interesting that the connection $\bar{\nabla}$ presented in (3.6) coincides with the connection $\tilde{\nabla}$ defined in [6]. In the following Lemma, we prove this equality

Lemma 3.4. *Let (M, g) be a Riemannian manifold with a 3-Sasakian structure. Then, the connection $\bar{\nabla}$ presented in (3.6) coincides with the connection $\tilde{\nabla}$ defined in* [6].

Proof. By the definition of $\tilde{\nabla}$ in [6] and using the local frame (3.1), we obtain

$$\tilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} = (\nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j})^h = F^k_{ij}\frac{\delta}{\delta x^k} = \bar{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j},$$

$$\tilde{\nabla}_{\xi_\alpha}\frac{\delta}{\delta x^i} = [\xi_\alpha, \frac{\delta}{\delta x^i}] = 0 = \bar{\nabla}_{\xi_\alpha}\frac{\delta}{\delta x^i},$$

$$\tilde{\nabla}_{\frac{\delta}{\delta x^i}}\xi_\alpha = \tilde{\nabla}_{\xi_\beta}\xi_\alpha = 0 = \bar{\nabla}_{\frac{\delta}{\delta x^i}}\xi_\alpha = \bar{\nabla}_{\xi_\beta}\xi_\alpha,$$

where $(\nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j})^h$ denotes the restricted component of the Levi-Civita connection ∇ on 3-contact distribution $\mathbf{H}$. □

From which, one can find the metrizability of $\bar{\nabla}$ and some more information about its torsion and curvature in [6]. However, we present the local expression of its torsion and curvature with respect to the frame (3.1) as follows

$$\begin{cases} T_{\bar{\nabla}}(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = 2\Omega^\alpha_{ij}\xi_\alpha, \\ T_{\bar{\nabla}}(\frac{\delta}{\delta x^i}, \xi_\alpha) = 0, \\ T_{\bar{\nabla}}(\xi_\alpha, \xi_\beta) = -T_{\bar{\nabla}}(\xi_\beta, \xi_\alpha) = -2\xi_\gamma, \end{cases} \tag{3.7}$$

for all even permutations (α, β, γ) of $(1, 2, 3)$.

$$\begin{cases} \bar{R}(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})\frac{\delta}{\delta x^k} = \bar{R}^h{}_{ijk}\frac{\delta}{\delta x^h}, \\ \bar{R}(\frac{\delta}{\delta x^i}, \xi_\alpha)\frac{\delta}{\delta x^j} = -\xi_\alpha(F^k_{ij})\frac{\delta}{\delta x^k}, \\ \bar{R}(\xi_\alpha, \xi_\beta)\frac{\delta}{\delta x^i} = 0, \\ \bar{R}(X, Y)\xi_\alpha = 0, \end{cases} \tag{3.8}$$

where $X, Y \in \Gamma(TM)$ and

$$\bar{R}^h{}_{ijk} = \frac{\delta F^h_{ij}}{\delta x^k} - \frac{\delta F^h_{ik}}{\delta x^j} + F^t_{ij}F^h_{tk} - F^t_{ik}F^h_{tj}. \tag{3.9}$$

Moreover, the Lie brackets of vector fields on M in terms of the $\mathbf{H}$-connection have the following expression

$$[X, Y] = \bar{\nabla}_X Y - \bar{\nabla}_Y X - 2\Omega^\alpha(X, Y)\xi_\alpha, \tag{3.10}$$

for any $X, Y \in \Gamma TM$.

Theorem 3.5. *Consider the linear connection $\bar{\nabla}$ given by (3.6) on 3-Sasakian manifold $(M, \varphi_\alpha, \eta^\alpha, \xi_\alpha, g)$ for $\alpha = 1, 2, 3$. Then, the following equation is satisfied*

$$(\bar{\nabla}_X\varphi_\alpha)Y = 0, \qquad \forall\alpha = 1, 2, 3$$

where $X, Y \in \Gamma\mathbf{H}$.

Proof. To complete the proof, we need to evaluate $(\bar{\nabla}_{\frac{\delta}{\delta x^i}}\varphi_\alpha)\frac{\delta}{\delta x^j}$. Using (3.6), we obtain

$$\begin{aligned}(\bar{\nabla}_{\frac{\delta}{\delta x^i}}\varphi_\alpha)\frac{\delta}{\delta x^j} &= \bar{\nabla}_{\frac{\delta}{\delta x^i}}\varphi_\alpha(\frac{\delta}{\delta x^j}) - \varphi_\alpha(\bar{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j}) \\ &= \nabla_{\frac{\delta}{\delta x^i}}\varphi_\alpha(\frac{\delta}{\delta x^j}) + \Omega^\beta(\frac{\delta}{\delta x^i}, \varphi_\alpha(\frac{\delta}{\delta x^j}))\xi_\beta - \varphi_\alpha(\nabla_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} + \Omega^\beta(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})\xi_\beta) \\ &= (\nabla_{\frac{\delta}{\delta x^i}}\varphi_\alpha)\frac{\delta}{\delta x^j} + \Omega^\beta(\frac{\delta}{\delta x^i}, \varphi_\alpha(\frac{\delta}{\delta x^j}))\xi_\beta - \Omega^\beta(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})\varphi_\alpha(\xi_\beta)\end{aligned}$$

$$= g(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})\xi_\alpha + g(\frac{\delta}{\delta x^i}, \varphi_1\varphi_\alpha(\frac{\delta}{\delta x^j}))\xi_1 - g(\frac{\delta}{\delta x^i}, \varphi_1(\frac{\delta}{\delta x^j}))\varphi_\alpha(\xi_1)$$
$$+g(\frac{\delta}{\delta x^i}, \varphi_2\varphi_\alpha(\frac{\delta}{\delta x^j}))\xi_2 - g(\frac{\delta}{\delta x^i}, \varphi_2(\frac{\delta}{\delta x^j}))\varphi_\alpha(\xi_2)$$
$$+g(\frac{\delta}{\delta x^i}, \varphi_3\varphi_\alpha(\frac{\delta}{\delta x^j}))\xi_3 - g(\frac{\delta}{\delta x^i}, \varphi_3(\frac{\delta}{\delta x^j}))\varphi_\alpha(\xi_3)$$

It is easy to check the last equation is vanish for all $\alpha = 1, 2, 3$. □

4. HyperKähler Contact Distribution and its Holomorphic Sectional Curvatures

If we restrict metric g and φ_α for $\alpha = 1, 2, 3$ to the 3-contact distribution $\mathbf{H}$, then $\mathbf{H}$ can be considered as an almost Hyper-Hermitian vector bundle. Moreover, φ_α for $\alpha = 1, 2, 3$ are parallel with respect to the metric connection $\bar{\nabla}$ on $\mathbf{H}$. Therefore, $\mathbf{H}$ carries an analogue HyperKähler structure and we call it a *HyperKähler contact distribution*. The close relation of HyperKähler and 3-Sasakian manifolds suggests this name as well.

To study the curvature tensor of 3-contact distribution $\mathbf{H}$, we present some properties of this tensor. By Corollary 3.2, it is deduced that g_{ij} are functions of (x^i) for $i = 1, ..., 4n$ (i.e. $\xi_\alpha(g_{ij}) = 0$ for all $\alpha = 1, 2, 3$). Using this fact and $[\frac{\delta}{\delta x^i}, \xi_\alpha] = 0$, one can check that $\xi_\alpha(F^k_{ij}) = 0$ and therefore the equations (3.8) and (3.9) become

$$\begin{cases} \bar{R}(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})\frac{\delta}{\delta x^k} = \bar{R}^h{}_{ijk}\frac{\delta}{\delta x^h}, \\ \bar{R}(\frac{\delta}{\delta x^i}, \xi_\alpha)\frac{\delta}{\delta x^j} = \bar{R}(\xi_\alpha, \xi_\beta)\frac{\delta}{\delta x^i} = \bar{R}(X, Y)\xi_\alpha = 0, \end{cases} \tag{4.1}$$

where $X, Y \in \Gamma(TM)$ and

$$\bar{R}^h{}_{ijk} = \frac{\partial F^h_{ij}}{\partial x^k} - \frac{\partial F^h_{ik}}{\partial x^j} + F^t_{ij}F^h_{tk} - F^t_{ik}F^h_{tj}. \tag{4.2}$$

The equations (4.1) and (4.2) show that $\bar{R}$ just depends on $\mathbf{H}$. Therefore, we put

$$\bar{R}(X, Y, Z, W) = g(\bar{R}(X, Y)W, Z), \quad \forall X, Y, Z, W \in \Gamma\mathbf{H}, \tag{4.3}$$

and we call it the *curvature tensor field* of $(\mathbf{H}, g|_{\mathbf{H}})$.

Lemma 4.1. *The curvature tensor field $\bar{R}$ of the HyperKähler contact distribution $\mathbf{H}$ satisfies the identities:*

$$\begin{array}{l} 1.\ \bar{R}(X, Y, Z, U) = -\bar{R}(Y, X, Z, U) = -\bar{R}(X, Y, U, Z), \\ 2.\ \bar{R}(X, Y, U, Z) + \bar{R}(Y, Z, U, X) + \bar{R}(Z, X, U, Y) = 0, \\ 3.\ \bar{R}(X, Y, Z, U) = \bar{R}(Z, U, X, Y), \\ 4.\ \bar{R}(X, Y, \varphi_\alpha Z, \varphi_\alpha U) = \bar{R}(X, Y, Z, U) = \bar{R}(\varphi_\alpha X, \varphi_\alpha Y, Z, U), \end{array} \tag{4.4}$$

for any $X, Y, Z, U \in \Gamma\mathbf{H}$.

Proof. The first equality in 1 is a general property of curvature tensor of any linear connection. The next equality of 1 is a consequence of the fact that $\bar{\nabla}$

is a metric connection. To prove 2, we use the Eq. (3.10) in a straightforward calculation as follows

$$\bar{R}(X,Y)Z+\bar{R}(Y,Z)X+\bar{R}(Z,X)Y=\bar{\nabla}_X\bar{\nabla}_YZ-\bar{\nabla}_Y\bar{\nabla}_XZ-\bar{\nabla}_{[X,Y]}Z$$
$$+\bar{\nabla}_Y\bar{\nabla}_ZX-\bar{\nabla}_Z\bar{\nabla}_YX-\bar{\nabla}_{[Y,Z]}X+\bar{\nabla}_Z\bar{\nabla}_XY-\bar{\nabla}_X\bar{\nabla}_ZY-\bar{\nabla}_{[Z,X]}Y$$
$$=[X,[Y,Z]]+2(\Omega^\alpha(X,[Y,Z])+\Omega^\alpha(\bar{\nabla}_XY,Z)+\Omega^\alpha(Y,\bar{\nabla}_XZ))\xi_\alpha$$
$$+[Y,[Z,X]]+2(\Omega^\alpha(Y,[Z,X])+\Omega^\alpha(\bar{\nabla}_YZ,X)+\Omega^\alpha(Z,\bar{\nabla}_YX))\xi_\alpha$$
$$+[Z,[X,Y]]+2(\Omega^\alpha(Z,[X,Y])+\Omega^\alpha(\bar{\nabla}_ZX,Y)+\Omega^\alpha(X,\bar{\nabla}_ZY))\xi_\alpha=0.$$

Then, by Lemma 3.1 in [18] on page 32 the item 3 is obtained. Finally, taking into account that φ_α are parallel with respect to $\bar{\nabla}$ on $\mathbf{H}$, the first equality of 4 is obtained. The next equality is achieved by means of the first one and item 3. □

First we have $\bar{R}(X,Y)Z$ in term of $R(X,Y)Z$, for all $X,Y,Z\in\Gamma(TM)$, as follows

$$\begin{aligned}\bar{\nabla}_X\bar{\nabla}_YZ=\nabla_X\nabla_YZ&\\ &+\eta^\alpha(X)\varphi_\alpha(\nabla_YZ)+\eta^\alpha(Y)\varphi_\alpha(\nabla_XZ)+\eta^\alpha(Z)\varphi_\alpha(\nabla_XY)\\ &+\eta^\alpha(\nabla_YZ)\varphi_\alpha(X)+\eta^\alpha(\nabla_XZ)\varphi_\alpha(Y)+\eta^\alpha(\nabla_XY)\varphi_\alpha(Z)\\ &+\Omega^\alpha(X,\nabla_YZ)\xi_\alpha+\Omega^\alpha(Y,\nabla_XZ)\xi_\alpha+\Omega^\alpha(\nabla_XY,Z)\xi_\alpha\\ &-(\Omega^\alpha(Y,X)\varphi_\alpha(Z)+\Omega^\alpha(Z,X)\varphi_\alpha(Y))\\ &+\eta^\alpha(Y)\eta^\beta(X)\varphi_\beta\varphi_\alpha(Z)+\eta^\alpha(Z)\eta^\beta(X)\varphi_\beta\varphi_\alpha(Y)\\ &-2\textstyle\sum_{\alpha=1}^3\eta^\alpha(Y)\eta^\alpha(Z)X+2\eta^\alpha(Y)g(X,Z)\xi_\alpha\\ &+\eta^\alpha(Y)\Omega^\beta(X,\varphi_\alpha(Z))\xi_\beta+\eta^\alpha(Z)\Omega^\beta(X,\varphi_\alpha(Y))\xi_\beta\\ &+\eta^\alpha(Y)\eta^\beta(\varphi_\alpha(Z))\varphi_\beta(X)+\eta^\alpha(Z)\eta^\beta(\varphi_\alpha(Y))\varphi_\beta(X),\end{aligned}\tag{4.5}$$

and

$$\begin{aligned}\bar{\nabla}_{[X,Y]}Z=\nabla_{[X,Y]}Z&\\ &+\eta^\alpha([X,Y])\varphi_\alpha(Z)+\eta^\alpha(Z)\varphi_\alpha([X,Y])+\Omega^\alpha([X,Y],Z)\xi_\alpha.\end{aligned}\tag{4.6}$$

Therefore,

$$\begin{aligned}\bar{R}(X,Y)Z=R(X,Y)Z&\\ &-2\Omega^\alpha(Y,X)\varphi_\alpha(Z)-\Omega^\alpha(Z,X)\varphi_\alpha(Y)+\Omega^\alpha(Z,Y)\varphi_\alpha(X)\\ &+\textstyle\sum_{\alpha=1}^3(\eta^\alpha(X)\eta^\alpha(Z)Y-\eta^\alpha(Y)\eta^\alpha(Z)X)\\ &-\eta^\alpha(Z)\eta^\beta(Y)\varphi_\beta\varphi_\alpha(X)+\eta^\alpha(Z)\eta^\beta(X)\varphi_\beta\varphi_\alpha(Y)\\ &+2\eta^\alpha(Y)\eta^\beta(X)\varphi_\beta\varphi_\alpha(Z)+2\eta^\alpha(Z)\Omega^\beta(X,\varphi_\alpha(Y))\xi_\beta\\ &-\eta^\alpha(X)\Omega^\beta(Y,\varphi_\alpha(Z))\xi_\beta+\eta^\alpha(Y)\Omega^\beta(X,\varphi_\alpha(Z))\xi_\beta\\ &+\eta^\alpha(Y)\eta^\beta(\varphi_\alpha(Z))\varphi_\beta(X)+\eta^\alpha(Z)\eta^\beta(\varphi_\alpha(Y))\varphi_\beta(X)\\ &-\eta^\alpha(X)\eta^\beta(\varphi_\alpha(Z))\varphi_\beta(Y)-\eta^\alpha(Z)\eta^\beta(\varphi_\alpha(X))\varphi_\beta(Y)\\ &+\eta^\alpha(Y)g(X,Z)\xi_\alpha-\eta^\alpha(X)g(Y,Z)\xi_\alpha,\end{aligned}\tag{4.7}$$

where the Einstein notation is used for repeated indices α and β on their range in case of $\alpha\neq\beta$.

Corollary 4.2. *Let M be a 3-Sasakian manifold and $(\mathbf{H}, g|_{\mathbf{H}})$ the HyperKähler contact distribution on M. Then the following holds for each vector field $X \in \Gamma\mathbf{H}$*

$$\bar{R}(X, \varphi_1 X, \varphi_2 X, \varphi_3 X) = R(X, \varphi_1 X, \varphi_2 X, \varphi_3 X).$$

The *Ricci tensor* $\bar{S}$ of the HyperKähler contact distribution $\mathbf{H}$ is defined by

$$\bar{S}(X, Y) = \sum_{i=1}^{4n} \bar{R}(v_i, X, v_i, Y) \quad \forall X, Y \in \Gamma\mathbf{H}, \tag{4.8}$$

where $\{v_1, ..., v_{4n}\}$ is an orthonormal local basis of vectors in $\Gamma\mathbf{H}$. Then, the local components of Ricci tensor $\bar{S}$ are given by

$$\bar{S}_{ij} = \bar{S}(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = R^k{}_{kij}. \tag{4.9}$$

Lemma 4.3. *Let M be a connected 3-Sasakian manifold of dimension $4n+3$, with $n > 1$. Then, the Ricci tensor $\bar{S}$ of the HyperKähler contact distribution $\mathbf{H}$ satisfies*

$$\bar{S}(X, Y) = (4n + 8)g(X, Y) \quad \forall X, Y \in \Gamma\mathbf{H}.$$

Proof. In [10], it was proved that 3-Sasakian manifolds are Einstein spaces and their Ricci tensor fields with respect to the Levi-Civita connection are given by

$$S(X, Y) = (4n + 2)g(X, Y) \quad \forall X, Y \in \Gamma TM,$$

where $dim(M) = 4n + 3$.
If $\{v_1, ..., v_{4n}\}$ is an orthonormal local basis of $\mathbf{H}$ then $\{\xi_1, \xi_2, \xi_3, v_1, ..., v_{4n}\}$ will be an unitary orthogonal local basis of TM, and vice versa. Using (2.2), (4.7) and (4.8), for all $X, Y \in \Gamma\mathbf{H}$, we obtain

$$\bar{S}(X, Y) = \sum_{i=1}^{4n} g(\bar{R}(v_i, X)Y, v_i) = \sum_{i=1}^{4n} g(R(v_i, X)Y, v_i)$$

$$-3\sum_{\alpha=1}^{3}\sum_{i=1}^{4n} g(X, \varphi_\alpha v_i)g(\varphi_\alpha Y, v_i) = S(X, Y) - \sum_{\alpha=1}^{3} g(R(\xi_\alpha, X)Y, \xi_\alpha) + 9g(X, Y)$$

$$= (4n + 2)g(X, Y) - 3g(X, Y) + 9g(X, Y) = (4n + 8)g(X, Y).$$

□

The sectional curvature K of Levi-Civita connection ∇ for the plane Π spanned by $\{X, Y\}$ at a point is given by

$$K(\Pi) = K(X, Y) = -\frac{R(X, Y, X, Y)}{g(X, X)g(Y, Y) - g^2(X, Y)}.$$

It is easy to check that $K(\Pi)$ is independent of choosing the vector fields X and Y spanned Π.
The *holomorphic sectional* curvatures of the HyperKähler contact distribution $\mathbf{H}$ are defined by

$$H_\alpha(X) = \bar{R}(X, \varphi_\alpha X, X, \varphi_\alpha X) \quad \forall \alpha = 1, 2, 3$$

where $X \in \Gamma\mathbf{H}$ has unit length at any point with respect to the metric g.

Definition 4.4. Let M be a 3-Sasakian manifold. The plane Π is called a φ_α-plane, whenever for any $X \in \Pi$, the sections X and $\varphi_\alpha X$ span Π.

Theorem 4.5. *Let M be a 3-Sasakian manifold of dimension $4n+3$, with $n > 1$. If only one of the holomorphic sectional curvatures of the HyperKähler contact distribution $\mathbf{H}$ does not depend on the respective φ_α-plane $\{X, \varphi_\alpha X\}$, then all holomorphic sectional curvatures are constant on M. Moreover, this constant value is equal to $\frac{4n+8}{n-2}$.*

Proof. First, without losing the generality, suppose that the holomorphic sectional curvature H_1 of $\mathbf{H}$ is independent of the $\{X, \varphi_1 X\}$-plane and it is given by $H_1(\Pi) = h$ where h is a function on M. Now, let

$$\begin{aligned} R_0(X,Y,Z,U) = \tfrac{1}{4}(&g(X,Z)g(Y,U) - g(X,U)g(Y,Z) \\ &+ \textstyle\sum_{\alpha=1}^3 g(X,\varphi_\alpha Z)g(Y,\varphi_\alpha U) - \sum_{\alpha=1}^3 g(X,\varphi_\alpha U)g(Y,\varphi_\alpha Z) \\ &+ 2\textstyle\sum_{\alpha=1}^3 g(X,\varphi_\alpha Y)g(Z,\varphi_\alpha U)) \qquad \forall X,Y,Z,U \in \Gamma\mathbf{H}. \end{aligned} \tag{4.10}$$

It is easy to check that R_0 satisfied (4.4). Moreover,

$$R_0(X, \varphi_\alpha X, X, \varphi_\alpha X) = 1, \qquad \alpha = 1,2,3 \tag{4.11}$$

for all unitary vector field $X \in \Gamma\mathbf{H}$. Therefore, we obtain

$$\bar{R}(X, \varphi_1 X, X, \varphi_1 X) = hR_0(X, \varphi_1 X, X, \varphi_1 X), \qquad \forall X \in \Gamma\mathbf{H}.$$

By lemma 4.1 on page 132 of [18], the following are obtained

$$\bar{R}(X,Y,Z,U) = hR_0(X,Y,Z,U), \qquad \forall X,Y,Z,U \in \Gamma\mathbf{H}. \tag{4.12}$$

Now, by calculating the Ricci tensor of R_0 and using lemma 4.3 we obtain

$$(4n+8)g(X,Y) = \bar{S}(X,Y) = h\sum_{i=1}^{4n} R_0(v_i, X, v_i, Y)$$

$$= \frac{h}{4}(4ng(X,Y) - g(X,Y) + \sum_{\alpha=1}^{3} g(X,Y) + 2\sum_{\alpha=1}^{3} g(X,Y)) = h(n-2)g(X,Y)$$

Therefore,

$$h = \frac{4n+8}{n-2}. \tag{4.13}$$

From (4.11)–(4.13), we conclude that

$$H_\alpha(X) = \bar{R}(X, \varphi_\alpha X, X, \varphi_\alpha X) = hR_0(X, \varphi_\alpha X, X, \varphi_\alpha X) = \frac{4n+8}{n-2},$$

where $\alpha = 1,2,3$ and X is an unitary vector field in $\mathbf{H}$. This completes the proof. $\square$

In theorem 4.5, it is proved that if a holomorphic sectional curvature of $\mathbf{H}$ is constant then all of them are constant and equal. In this case, we call $\mathbf{H}$ a HyperKähler contact distribution of *constant holomorphic sectional* curvature. By means of this definition and theorem 4.5 the following corollary is obtained

Corollary 4.6. *Let M be a 3-Sasakian manifold of dimension $4n+3$, with $n > 1$. Then the HyperKähler contact distribution $\mathbf{H}$ is of constant holomorphic sectional curvature $\frac{4n+8}{n-2}$ if and only if its curvature tensor field $\bar{R}$ on $\mathbf{H}$ is given by*

$$\bar{R}(X,Y)Z = \frac{k}{4}(g(Y,Z)X - g(X,Z)Y + \sum_{\alpha=1}^{3} g(\varphi_\alpha Y, Z)\varphi_\alpha X$$

$$-\sum_{\alpha=1}^{3} g(\varphi_\alpha X, Z)\varphi_\alpha Y + 2\sum_{\alpha=1}^{3} g(X, \varphi_\alpha Y)\varphi_\alpha Z) \qquad \forall X,Y,Z \in \Gamma\mathbf{H}.$$

Theorem 4.7. *Let M be a 3-Sasakian manifold with HyperKähler contact distribution $\mathbf{H}$. Then $\mathbf{H}$ is of constant holomorphic sectional curvature k if and only if any φ_α-sectional curvature of M for a φ_α-plane $\{X, \varphi_\alpha X\}$, where $X \in \Gamma\mathbf{H}$, is constant $k-3$ for $\alpha = 1,2,3$.*

Proof. It is a straightforward consequence of (4.7). □

Up to now, we use metric connection $\bar{\nabla}$ on HyperKähler contact distribution $\mathbf{H}$ to define the curvature tensor, holomorphic sectional curvatures and Ricci tensor of this distribution. Moreover, in some theorems, we present them in terms of curvature tensor, φ_α-sectional curvatures and Ricci tensor of Levi-Civita connection on $\mathbf{H}$, respectively. In the next theorem, we study the sectional curvatures of these two connections on M.

Theorem 4.8. *Let Π be the φ_α-plane at a point of a 3-Sasakian manifold M. Then, the sectional curvatures of Π with respect to the ∇ and $\bar{\nabla}$ are related as follows*

$$\begin{aligned}\bar{K}(\Pi) = K(\Pi) &+ 3 + 4\left(\eta^\beta(X)\eta^\gamma(X)\right)^2 \\ &+6\left((\eta^\beta(X))^4 + (\eta^\gamma(X))^4\right) - 8\left((\eta^\beta(X))^2 + (\eta^\gamma(X))^2\right),\end{aligned}$$

where X is a unit vector in the φ_α-plane Π and (α,β,γ) is an arbitrary permutation of $(1,2,3)$.

Proof. Without losing the generality, we prove these theorem for $\alpha = 1$. Consider the α_1-plane Π and unit vector $X \in \Pi$, then by using (4.7), we obtain

$$\bar{K}(\Pi) = \bar{R}(X,\varphi_1 X, X, \varphi_1 X) = R(X,\varphi_1 X, X, \varphi_1 X)$$

$$-(2+1)\Omega^\beta(\varphi_1 X, X)g(\varphi_\beta\varphi_1 X, X) + \Omega^\beta(\varphi_1 X, \varphi_1 X)g(\varphi_\beta X, X)$$

$$+\sum_{\beta=1}^{3}\left(\eta^\beta(X)\eta^\beta(\varphi_1 X)g(\varphi_1 X, X) - \eta^\beta(\varphi_1 X)\eta^\beta(\varphi_1 X)g(X,X)\right)$$

$$-\eta^\beta(\varphi_1 X)\eta^\gamma(\varphi_1 X)g(\varphi_\gamma\varphi_\beta X, X) + (1+2)\eta^\beta(\varphi_1 X)\eta^\gamma(X)g(\varphi_\gamma\varphi_\beta\varphi_1 X, X)$$

$$+(1+1)\eta^\beta(\varphi_1 X)\eta^\gamma(\varphi_\beta\varphi_1 X)g(\varphi_\gamma X, X) - \eta^\beta(X)\eta^\gamma(\varphi_\beta\varphi_1 X)g(\varphi_\gamma\varphi_1 X, X)$$

$$-\eta^\beta(\varphi_1 X)\eta^\gamma(\varphi_\beta X)g(\varphi_\gamma\varphi_1 X, X) + \eta^\beta(\varphi_1 X)g(X,\varphi_1 X)g(\xi_\beta, X)$$

$$-\eta^\beta(X)g(\varphi_1 X, \varphi_1 X)g(\xi_\beta, X) + (2+1)\eta^\beta(\varphi_1 X)\Omega^\gamma(X, \varphi_\beta\varphi_1 X)g(\xi_\gamma, X)$$

$$-\eta^\beta(X)\Omega^\gamma(\varphi_1 X, \varphi_\beta\varphi_1 X)g(\xi_\gamma, X) = K(\Pi) + 3 + 4\left(\eta^\beta(X)\eta^\gamma(X)\right)^2$$

$$+6\left((\eta^{\beta}(X))^4+(\eta^{\gamma}(X))^4\right)-8\left((\eta^{\beta}(X))^2+(\eta^{\gamma}(X))^2\right).$$

This completes the proof. □

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